5.1 – Midsegment Theorem

A midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. Every triangle has three midsegments. In triangle ABC below, the midsegments are MP, MN and NP.

Theorem 5.1 – Midsegment Theorem
The segment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle and is half as long.

\[ DE \parallel AC \text{ and } DE = \frac{1}{2} AC \]

5.2 – Perpendicular Bisectors

Theorem 5.2 – Perp. Bisector Theorem
In a plane, if a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

In the figure below, if \( CP \) is the \( \perp \) bisector of \( AB \), then \( CA = CB \).

Theorem 5.3 – Perp. Bisector Converse
In a plane, if a point is equidistant from the endpoints of the segment, then it is on the perpendicular bisector of the segment.

In the figure below, if \( DA = DB \), then \( D \) lies on the \( \perp \) bisector of \( AB \).

Concurrency – When three or more lines, rays or segments intersect in the same point, they are called concurrent. The point of intersection is called a point of concurrency.

Theorem 5.4 – Concurrency of Perpendicular Bisectors
The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

In the figure below, if \( PD, PE \) and \( PF \) are perpendicular bisectors, then \( PA = PB = PC \).

Circumcenter – The point of concurrency of the three perpendicular bisectors of a triangle is called the circumcenter of the triangle.

Location of Circumcenter
Acute triangle – inside the triangle
Right triangle – on the triangle
Obtuse triangle – outside the triangle
### 5.3 – Angle Bisectors

**Theorem 5.5 – Angle Bisector Theorem**

If a point is on the bisector of an angle, then it is equidistant from the sides of the angle.

In the figure below, if $\overline{AD}$ bisects $\angle BAC$, $DB \perp AB$ and $DC \perp AC$, then $CA = CB$.

**Theorem 5.6 – Angle Bisector Converse**

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it is on the bisector of an angle.

In the figure below, if $DB \perp AB$, $DC \perp AC$ and $DB = DC$, then $\overline{AD}$ bisects angle $BAC$.

**Theorem 5.7 – Concurrency of Angle Bisectors**

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

In the figure below, if $\overline{AP}$, $\overline{BP}$ and $\overline{CP}$ are angle bisectors, then $PD = PE = PF$.

**Incenter** – The point of concurrency of the three angle bisectors of a triangle is called the incenter of the triangle. The incenter always lies inside the triangle.

### 5.4 – Medians and Altitudes

**Median** – A median of a triangle is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are concurrent. The point of concurrency, called the centroid, is inside the triangle.

**Theorem 5.8 – Concurrency of Medians**

The medians of a triangle intersect at a point that is two-thirds of the distance from each vertex to the midpoint of the opposite side.

In the figure below, if $AE$, $BF$ and $CD$ are angle bisectors, then $AP = 2*PE$, $BP = 2*PF$ and $CP = 2*PD$.

**Altitudes** – An altitude of a triangle is a perpendicular segment from a vertex to the opposite side or to a line that contains the opposite side. The three altitudes of a triangle are concurrent. The point of concurrency is called the orthocenter.

**Theorem 5.9 – Concurrency of Altitudes**

The lines containing the altitudes of a triangle are concurrent.

**Location of Orthocenter**

Acute triangle – inside the triangle
Right triangle – on the triangle
Obtuse triangle – outside the triangle
5.5 – Inequalities in a Triangle

**Theorem 5.10 – Unequal Sides**
If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side. 

![Diagram of a triangle with sides labeled A, B, and C.](image)

\[ AB > BC, \text{ so } m\angle C > m\angle A. \]

**Theorem 5.11 – Unequal Angles**
If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

![Diagram of a triangle with angles labeled A, B, and C.](image)

\[ m\angle A > m\angle C, \text{ so } BC > AB. \]

**In summary**, the longest side of any triangle is always opposite the largest angle of the triangle and vice-versa. Likewise, the shortest side of any triangle is always opposite the smallest angle of the triangle and vice-versa.

**Theorem 5.12 – Triangle Inequality Thm.**
The sum of the lengths of any 2 sides of a triangle is greater than the third side.

In the triangle below this means that \( AB+BC>AC, \ AC+BC>AB, \) and \( AB+AC>BC, \)

![Diagram of a triangle with sides labeled A, B, and C.](image)

Possible Side Lengths in a Triangle
If two sides of a triangle have sides of lengths \( A \) and \( B \) \((B > A)\), then the length of the third side \((\text{call it } C)\) must be between \( B-A \) and \( B+A \). So, \( B-A < C < B+A \).

**Example:**
Suppose two sides of a triangle are 5 and 12. Since \( 12 - 5 = 7 \) and \( 12 + 5 = 17 \), the length of the third side must be between 7 and 17.

5.6 – Inequalities in Two Triangles

**Theorem 5.13 – Hinge Theorem**
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first triangle is larger than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle.

![Diagram of two triangles with sides labeled V, W, X, Y, Z, and angles labeled 88°, 35°, and 12°.](image)

\[ WX > ST \]

**Theorem 5.14 – Hinge Theorem Converse**
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is larger than the included angle of the second.

![Diagram of two triangles with sides labeled 12 and 9.](image)

\[ m\angle C > m\angle F \]