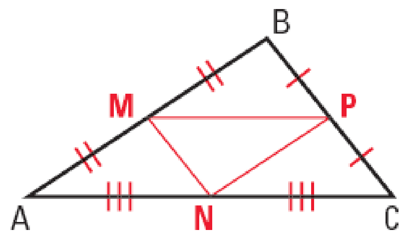
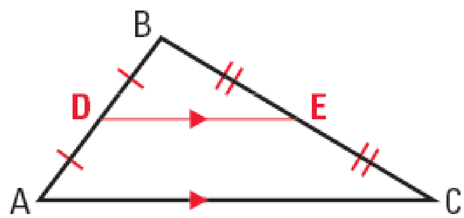
5.1 – Midsegment Theorem

A **midsegment of a triangle** is a segment that connects the midpoints of two sides of the triangle. Every triangle has three midsegments. In triangle ABC below, the midsegments are MP, MN and NP.



Theorem 5.1 – Midsegment Theorem
The segment connecting the midpoints of two sides of a triangle is parallel to the third side of the triangle and is half as long.

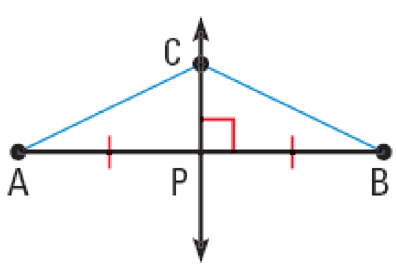


$\overline{DE} \parallel \overline{AC}$ and $DE = \frac{1}{2}AC$

5.2 – Perpendicular Bisectors

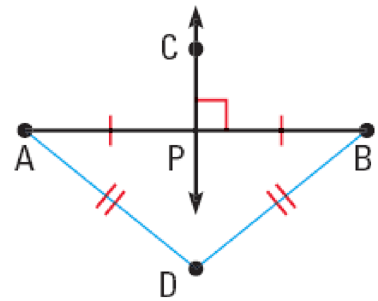
Theorem 5.2 – Perp. Bisector Theorem
In a plane, if a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

In the figure below, if $\overline{CP}$ is the $\perp$ bisector of AB, then CA = CB.



Theorem 5.3 – Perp. Bisector Converse
In a plane, if a point is equidistant from the endpoints of the segment, then it is on the perpendicular bisector of the segment,

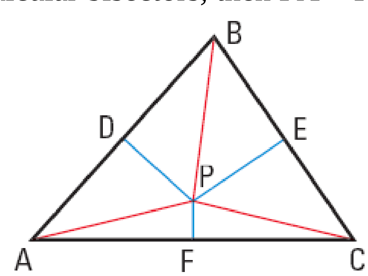
In the figure below, if DA = DB, then D lies on the $\perp$ bisector of $\overline{AB}$.



Concurrency – When three or more lines, rays or segments intersect in the same point, they are called **concurrent**. The point of intersection is called a point of **concurrency**.

Theorem 5.4 – Concurrency of Perpendicular Bisectors
The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

In the figure below, if $\overline{PD}$, $\overline{PE}$ and $\overline{PF}$ are perpendicular bisectors, then PA = PB = PC.



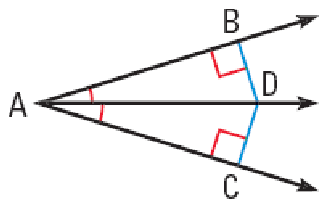
Circumcenter – The point of concurrency of the three perpendicular bisectors of a triangle is called the **circumcenter** of the triangle.

Location of Circumcenter
Acute triangle – inside the triangle
Right triangle – on the triangle
Obtuse triangle – outside the triangle

5.3 – Angle Bisectors

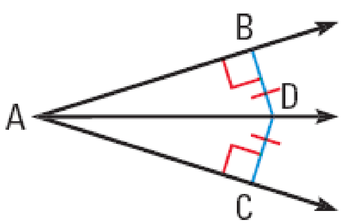
Theorem 5.5 – Angle Bisector Theorem
If a point is on the bisector of an angle, then it is equidistant from the sides of the angle.

In the figure below, if $\overrightarrow{AD}$ bisects $\angle BAC$, $\overline{DB} \perp \overline{AB}$ and $\overline{DC} \perp \overline{AC}$, then CA = CB.



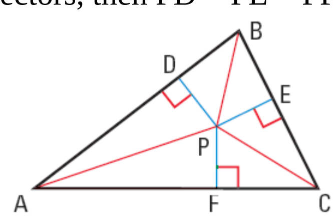
Theorem 5.6 – Angle Bisector Converse
If a point is in the interior of an angle and is equidistant from the sides of the angle, then it is on the bisector of an angle.

In the figure below, if $\overline{DB} \perp \overline{AB}$, $\overline{DC} \perp \overline{AC}$ and DB = DC, then $\overrightarrow{AD}$ bisects angle BAC.



Theorem 5.7 – Concurrency of Angle Bisectors
The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

In the figure below, if $\overrightarrow{AP}$, $\overrightarrow{BP}$ and $\overrightarrow{CP}$ are angle bisectors, then PD = PE = PF.



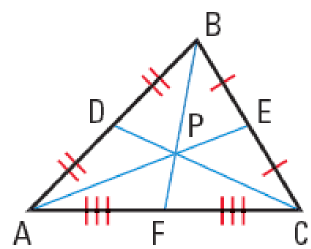
Incenter – The point of concurrency of the three angle bisectors of a triangle is called the **incenter** of the triangle. The incenter always lies inside the triangle

5.4 – Medians and Altitudes

Median – A **median of a triangle** is a segment from a vertex to the midpoint of the opposite side. The three medians of a triangle are *concurrent*. The point of concurrency, called the **centroid**, is inside the triangle

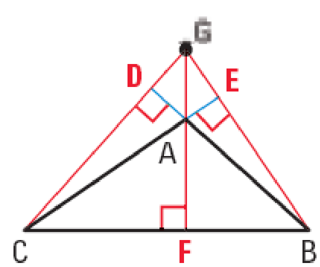
Theorem 5.8 – Concurrency of Medians
The medians of a triangle intersect at a point that is two-thirds of the distance from each vertex to the midpoint of the opposite side.

In the figure below, if $\overline{AE}$, $\overline{BF}$ and $\overline{CD}$ are angle bisectors, then AP = 2*PE, BP = 2*PF and CP = 2*PD.



Altitudes – An **altitude of a triangle** is a perpendicular segment from a vertex to the opposite side or to a line that contains the opposite side. The three altitudes of a triangle are *concurrent*. The point of concurrency is called the **orthocenter**.

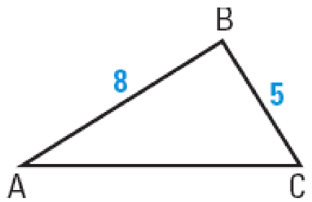
Theorem 5.9 – Concurrency of Altitudes
The lines containing the altitudes of a triangle are concurrent.



Location of Orthocenter
Acute triangle – inside the triangle
Right triangle – on the triangle
Obtuse triangle – outside the triangle

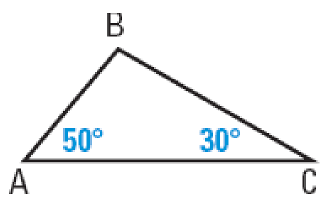
5.5 – Inequalities in a Triangle

Theorem 5.10 – Unequal Sides
If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.



$AB > BC$, so $m\angle C > m\angle A$.

Theorem 5.11 – Unequal Angles
If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

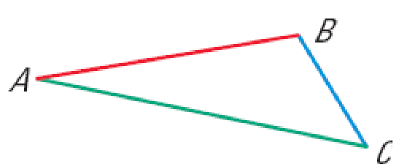


$m\angle A > m\angle C$, so $BC > AB$.

In summary, the longest side of any triangle is always opposite the largest angle of the triangle and vice-versa. Likewise, the shortest side of any triangle is always opposite the smallest angle of the triangle and vice-versa

Theorem 5.12 – Triangle Inequality Thm.
The sum of the lengths of any 2 sides of a triangle is greater than the third side.

In the triangle below this means that AB+BC>AC, AC+BC>AB, and AB+AC>BC,

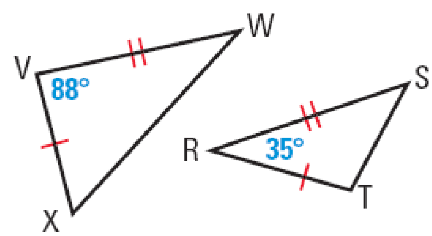


Possible Side Lengths in a Triangle
If two sides of a triangle have sides of lengths A and B (B > A), then the length of the third side (call it C) must be between $B-A$ and $B+A$. So, $B-A<C<B+A$.

Example:
Suppose two sides of a triangle are 5 and 12. Since 12 – 5 = 7 and 12 + 5 = 17, the length of the third side must be between 7 and 17.

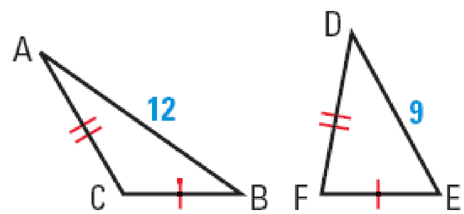
5.6 – Inequalities in Two Triangles

Theorem 5.13 – Hinge Theorem
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first triangle is larger than the included angle of the second, then the third side of the first triangle is longer than the third side of the second triangle.



$WX > ST$

Theorem 5.14 – Hinge Theorem Converse
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is larger than the included angle of the second.



$m\angle C > m\angle F$