Probability: The Study of Randomness – Counting Methods

Introduction

Suppose that a room full of 10 people all greet each other by fist-bumping. If every person bumps fists with every other person exactly once, how many fist bumps will take place? Let's explore!

In order to answer big questions we sometimes start small and work our way up. We can find patterns in the small problems and use them to solve the big problem. Here we want to find out the number of fist bumps for 10 people. Let's start by looking at the number of fist bumps needed for 2, 3, 4 and 5 people. We will then use our evidence to answer the bigger question as well as determine a solution for all such problems.

Let's start with two people. They can only make one fist bump between them. How about three people? To answer this, let's call the three people by the names A, B and C. So to illustrate the different fist bumps, A can fist bump with B as well as with C. B can also fist bump with C. That is all – so three people will need three total fist bumps. Let's continue this method as we look at the results for 4 and 5 people.

In the table below, we will again use letters to represent the people. To symbolize a fist bump between two people, we will use the two letters of the "names" of the people. The notation *AB*, for example, will represent a fist bump between person *A* and person *B*. A systematic list will then be made to show all possible fist bumps.

#	People	Fist Bumps	Count
2	A, B	AB	1
3	A, B, C	AB, AC, BC	3
4	A, B, C, D	AB, AC, AD, BC, BD, CD	6
5	A, B, C, D, E	AB, AC, AD, AE, BC, BD, BE, CD, CE, DE	10
6	A, B, C, D, E, F	?, …	?

Is there a pattern present that will allow us to correctly predict the number of fist bumps for 6 people? We can see that the number of fist bumps increases by 2 (from 1 to 3), then by 3 (from 3 to 6), then by 4 (from 6 to 10). Following this pattern, we can predict that the next number will increase by 5, from 10 to 15, meaning that 6 people will need 15 total fist bumps. Make a list like those in the table to verify this prediction.

Now back to our original question about the number of fist bumps between 10 people. Continuing the pattern observed above, 7 people would need 15 + 6 = 21 fist bumps. Eight people would need 21 + 7 = 28; nine people would need 28 + 8 = 36; and, finally, ten would require 36 + 9 = 45 fist bumps.

Our work is done here – or is it? How about a general formula that would enable us to find the total number of fist bumps needed for any give number of people?

For this we will start by calling the number of people we have by a variable, say *n*. So we will try to find a formula involving *n* that will allow us to directly determine the number of fist bumps without having to make a list as we did earlier. This will make it much easier to answer a question such as how many fist bumps will be needed for 100 people.

For *n* people, there will be $\frac{n(n-1)}{2}$ fist bumps. To see that this formula works for ten people, we calculate

$$\frac{10(9)}{2}=\frac{90}{2}=45$$

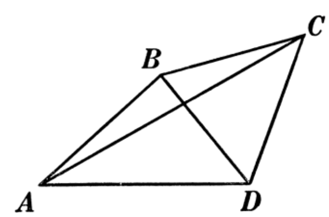
Recall that is what we had determined earlier. Taking this further, the number of fist bumps for a classroom full of 36 students would be

$$\frac{36(35)}{2}=\frac{1260}{2}=630$$

That's a lot of fist bumps! Now, here is another problem involving counting:

Given the number of sides of a polygon, how many different diagonals can be drawn? A diagonal of a polygon is a line segment that can be drawn to connect two nonconsecutive vertices (or points) of the polygon. This cannot be done with a triangle (a 3-sided polygon) and the minimum number of sides we must start with is 4 (a quadrilateral). See the diagram at the right below. Quadrilateral ABCD has two diagonals drawn. Complete the table below by sketching your own polygons of the given sizes and all diagonals possible. Count the number of diagonals and use the results to find a pattern.

Sides	Polygon Name	Number of Diagonals
4	quadrilateral	2
5	pentagon	
6	hexagon	
7	heptagon	



Based on the evidence, how many diagonals are possible in an octagon (an 8-sided polygon)? How many for a nonagon (9 sides)? A decagon (10 sides)? Finally, how about an *n*-gon (*n* sides)? **Hint**: look at the formula for the fist bumps problem and see if you can come up with a similar one that describes the pattern for the diagonals problem

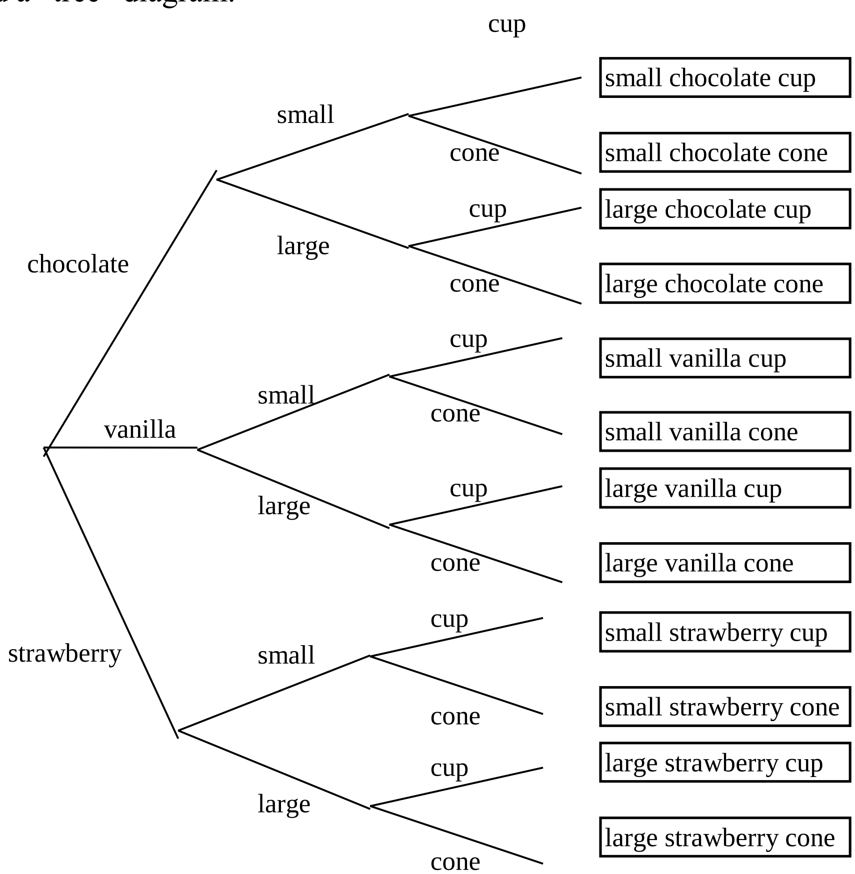
In Conclusion

These problems fall within a branch of mathematics called *combinatorics* that is concerned with determining the number of different ways that an event can take place. In this first lesson of our unit on probability, we will explore what are called *counting methods* and develop and use formulas that provide simple answers to complicated questions like the ones we just encountered on fist bumps and diagonals.

The Fundamental Counting Principle

Baskin Robbins, aka *31 Flavors*, began selling ice cream in 1953. They initially sold 31 flavors so that you could have a different flavor every day of the month. Let's take a trip to an ice cream parlor that has only 3 flavors: *chocolate*, *vanilla* and *strawberry*. Scoops come in two sizes, *small* and *large*. Also, you can have your scoop in a *cup* or a *cone*. What will you order?

How many different orders are possible at this store? Let's analyze the situation with what is called a "tree" diagram.



Our tree shows that there are 12 different possible orders. This is because there are 3 flavors, 2 sizes and 2 containers. The result is 3 x 2 x 2 = 12 possibilities, as is shown in the diagram. This is called the *Fundamental Counting Principle*. It says that if you want to find the number of overall outcomes with several parts, we just need to multiply the number of possibilities for each part.

The Fundamental Counting Principle

If one event can occur in a ways, and another event can occur in b ways, then together these events can occur in $a \times b$ ways. This concept can be extended to three or more events (for example, $a \times b \times c$).

EXAMPLES – Let's look at some more uses of the Fundamental Counting Principle.

Example 1: Deli-cious A sandwich at a deli can be made with white, wheat or rye bread. For meat, you can choose from ham, turkey, pastrami or roast beef. Lastly, you can choose from the following cheeses: Swiss, Muenster, cheddar, provolone or Pepper Jack. How many different types of sandwiches can be ordered from this deli?

Solution: There are 3 bread choices, 4 meats and 5 cheeses. So, there are $3 \times 4 \times 5 =$ **60** different sandwiches that can be made at this deli.

Example 2: Monopolize Many board games, like Monopoly, require players to roll two 6-sided dice to determine the number of spaces to move around the board on each turn. How many different dice rolls are possible in this situation?

Solution: Since there are two dice and each has 6 sides, there are $6 \times 6 =$ **36** different possible dice rolls in games like Monopoly.

Example 3: Lights Suppose an apartment has 8 light switches scattered throughout the different rooms in the dwelling. Considering that each switch can either be turn ON or OFF at any given time, how many different configurations of light switches are possible?

Solution: Since there are 8 light switches and each switch has 2 possible positions, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 =$ **256** different possible light switch configurations.

Example 4: Take Flight Airports around the world are identified by a unique, 3-letter code. For example, Los Angeles International Airport has the code LAX. Toronto airport is assigned the code YYZ. How many different airport codes are possible using 3 letters?

Solution: Since there are 26 letters in our alphabet, there are $26 \times 26 \times 26 =$ **17,576** different airport codes possible using 3 letters.

Example 5: Take Flight – the sequel! What if the airport codes described in Example 4 do not allow duplicate letters? That would mean Toronto's YYZ could not be used, but the code LAX for Los Angeles would be acceptable. With this revised rule, fewer codes will now be available. How many 3-letter codes would now be available?

Solution: Since there are 26 letters in our alphabet, there are still 26 choices for the first letter. However, the second letter can now only be chosen from the remaining 25 (one letter is already taken). Likewise, the third letter can only be chosen from the 24 left after the first two letters are taken. So, that means there are $26 \times 25 \times 24 =$ **15,600** different airport codes possible using 3 letters without allowing duplicates. This is 1,976 fewer airport codes than the previous answer (17,576 – 15,600 = 1,976).

Permutations

Suppose we decide to rearrange the 36 students in a class so that every day, they are in a different seating order. How many school days will this take?

As we did in the fist bump problem, let's start small and use patterns to work toward an answer to this larger question.

The table below shows the different orders for 1, 2 and 3 people. The list for 4 people has been started – complete it. As we did before with the fist bump problem, we use letters to represent different people.

People	List of different orders	Number of Orders
1	A	1
2	AB, BA	2
3	ABC, ACB, BAC, BCA, CAB, CBA	6
4	ABCD, ABDC, ACBD, ACDB, ADBC, ADCB BACD, BADC, BCAD, BCDA, BDAC, BDCA	
5	Too many to list here!	Prediction =

So what is the pattern? How can we use it to get the number of orders for 5 people?

Here is one way to look at the pattern – complete it for 4 and 5 people:

People	Number of Orders	Calculation
1	1	1 = 1
2	2	1 × 2 = 2
3	6	1 × 2 × 3 = 6
4		
5		

Why does this pattern work out this way? The answer lies, once again, in the Fundamental Counting Principle. This is the type of problem we saw earlier where our choices continually decrease when we can't use repeats. So, when we consider the number of different orders that 3 people can take, the first position has 3 choices, the second has 2 and the third has only 1. Which means that the number of arrangements is $3\times2\times1=6$. Note that this is the reverse order of the multiplication that we saw in the table above, but it still results in the same answer – that's the commutative property.

The type of calculation we just did has a special name and symbol in mathematics. The symbol is the exclamation point and the name is *factorial*. So we write $3\times2\times1$ as 3! and read it as "3 factorial." In general the factorial of any number *n* is called "*n factorial*" or *n*! and we compute it as $n!=n\cdot(n-1)\cdot(n-2)\cdots2\cdot1$.

Factorials are used in many types of counting problems. The type of counting problem we addressed here is one of *permutations*. A **permutation** is an ordered arrangement of items when the order of the items matters. When we rearrange all of a given collection of items in different orders, we will call those *simple* permutations.

Now let's formalize a general rule for any simple permutation problem.

Factorial Rule for Simple Permutations

A collection of *n* different items can be arranged in *n*! different orders where $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$

EXAMPLES – Let's look at more uses of the Factorial Rule for Simple Permutations

Example 1: Coasters You are visiting Six Flag's Magic Mountain and you decide to ride your favorite roller coasters once each: Goliath, Ninja, Scream, Tatsu, Viper and X2. You are making your plans for the order in which you will ride them. How many different orders could you choose from?

Solution: Since there are 6 coasters on your list, there are 6! different orders. So there are $6! = 6\times5\times4\times3\times2\times1 = 720$ different orders in which you can ride these coasters.

Example 2: On your marks! There are 8 runners lined up to run a 400-meter race (one time around the track). How many different orders could they finish in?

Solution: Since there are 8 runners in the race, there are 8! different orders. So there are $8! = 8\times7\times6\times5\times4\times3\times2\times1 = 40,320$ different orders in which they can finish the race.

Now let's look at a different version of the race problem in Example 2 above. Suppose that in our race with 8 runners, medals are awarded to the 1^{st} three finishers (so we have 1^{st} place, 2^{nd} place and 3^{rd} place medals). In how many different orders could we get 3 medalists out of 8 participants?

To answer this question, we can again refer to the Fundamental Counting Principle. Since there are 8 runners, there are 8 different people who could finish first. That leaves 7 runners who could finish 2^{nd} and then 6 runners who could finish 3^{rd}. All told then, the number of possible orders will be $8\cdot7\cdot6 = 336$ 8 x 7 x 6 = 336. So there are 336 different orders of 3 medalists possible from the 8 participants.

This is a different type of permutation – one in which we are looking to find the number of ways to arrange fewer than all of the *n* items. If we have *n* total items and we call the number of items we are interested in arranging *r*, then what we say is that we need the number of permutations of *r* items out of *n* items. Since this is a modified version of our simple permutation scenario, the formula and calculation will be a modified version of the factorial rule just discussed. Let's see how the formula changes.

In the solution to the 3 medalists out of 8 runners problem, we calculated the number of orders as $8 \cdot 7 \cdot 6 = 336$. That calculation is just the beginning of 8! It stops at the 6, which means we left out the $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ part. Another way to consider this calculation can be done using the following fraction:

$$8 \cdot 7 \cdot 6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

The fraction is equivalent to $8 \cdot 7 \cdot 6$ because we can cancel out the $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ that appears in both the numerator and denominator of the fraction which leaves just $8 \cdot 7 \cdot 6$.

Take note now that $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8!}{5!}$ so the calculation for the number of permutations of 3 items taken from 8 total items is simply $\frac{8!}{5!}$. This result will enable us to develop a general formula for the number of permutations of *r* items out of *n* items.

Take another example to make it clear. Consider the same race of 8 runners in which we now award medals for only 1st and 2nd place. So we would now need to know the number of permutations of 2 medalists from 8 racers. This would result in the calculation

$$8 \cdot 7 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8!}{6!} = 56$$

So there are 56 different possible orders of 2 medalists out of 8 race participants.

Let's summarize the results and find a pattern in a table:

Runners	**Medalists**	**Permutations**
8	3	$\frac{8!}{5!} = \frac{8!}{(8-3)!}$
8	2	$\frac{8!}{6!} = \frac{8!}{(8-2)!}$

So, we see that the numerator in the calculation is just the factorial of the number of total runners. The denominator is the factorial of the *difference* between the number of runners and the number of medalists. To generalize this scenario and arrive at a formula for any such problem, let *n* = the total number of runners, and *r* = the number of medalists. We can then create a formula for the number of permutations of *r* items out of *n* total items.

Rule for Permutations when all items are different

The number of permutations (or orders) of *r* items selected from a collection of *n* different available items (without replacement) is noted and calculated as

$$_nP_r = \frac{n!}{(n-r)!}$$

EXAMPLES – Let's look at more uses of the Rule for Permutations with Different Items

Example 1: Exacta-ly! In horse racing, there is a bet called an *exacta*. With this type of bet, you have to correctly pick the 1^{st} and 2^{nd} place finishers in the race. Suppose there are 12 horses in a given race. How many different ways are there to select an exacta bet?

Solution: The order of finish matters here, so this is a permutations calculation. Since there are 12 horses and a bettor must select the first 2 to finish the race, this is a calculation of the form ${}_{12}P_2$. We calculate the result as follows:

$$\frac{12!}{(12-2)!} = \frac{12!}{10!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 12 \cdot 11 = 132$$

So there are 132 different ways to place an exacta bet in a race with 12 horses.

Example 2: Science Clubbin' The Science Club has 9 members who would like to be club officers. There are 4 positions available: President, Vice-President, Secretary and Treasurer? If the officers are chosen by randomly selecting 4 of the 9 available members, how many different ways can they fill the positions?

Solution: Order again matters here as each person selected will have a different position. Since there are 9 available members and they will select 4, this is a permutations calculation of the form ${}_9P_4$. We calculate the result as follows:

$$\frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$$

So there are 3,024 different orders in which they can select their 4 officers.

Example 3: Raffling! A charity organization is conducting a raffle in which 100 tickets will be sold at $20 each. After the tickets have been also sold, there will be a drawing to randomly select 3 winners of the prizes. The first prize is a television; second prize is a video game system and the third prize is a portable mp3 music player. How many different ways can they choose the three prize winners from the 100 ticket buyers?

Solution: It is important to note that the order of the tickets being drawn makes this a permutations problem – each person gets a different prize. Now, since there are 100 available ticket buyers and they will select 3, this is a permutations calculation of the form ${}_{100}P_3$. We calculate the result as follows:

$$\frac{100!}{(100-3)!} = \frac{100!}{97!} = 100 * 99 * 98 = 970,200$$

So there are 970,200 different ways to select the three prize winners from the 100 tickets.

Combinations

Let's take a different look at the Science Club example from the previous section. In that example, 4 people were being chosen out of a group of 9 to fill specific positions. Now suppose that 4 of the club members are being chosen to represent the club at a student council meeting. In this situation, it does not matter which member is chosen first, second, third or fourth. So the order of the random selection does not matter. All four will go to the student council meeting regardless of when they were chosen.

So this is a different counting problem – not a permutation. When selecting from a group when order does not matter, we call this a combination.

Rule for Combinations

The number of combinations of *r* items selected from a collection of *n* different available items (without replacement) is noted and calculated as

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

EXAMPLES – Let's look at more uses of the Rule for Combinations

Example 3: A Lotto Fun! In 1986, the state of California began operating a lottery game called the *Lotto 6/49* in which players paid $1 to select 6 different numbers in the range of 1 to 49. On Saturday nights, the state gaming commission would televise the random selection of 6 numbers. If a player matched all 6 of the chosen numbers, they would win the grand prize which was usually several millions dollars. Note that the order of chosen numbers does not matter. The winner simply has to have the same 6 numbers. How many different ways are there to select 6 numbers out of the numbers 1 to 49?

Solution: Since order does not matter, this a combinations calculation, not one of permutations. Since there are 49 numbers and a player must select 6 of them, this is a calculation of the form ${}_{49}C_6$. We calculate the result as follows:

$$\frac{49!}{6!(49-6)!} = \frac{49}{6!43!} = 13{,}983{,}816$$

So there are 13,983,816 different ways to select 6 numbers from the numbers 1 to 49.

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$$\frac{9!}{(9-4)!}=\frac{9!}{5!}=\frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1}=9\cdot 8\cdot 7\cdot 6=3,024$$

So there are 3,024 different orders in which they can select their 4 officers.

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